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Comparison theorems for even order half-linear differential equations

Jaroslav Jaroš

Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University, 842 48 Bratislava, Slovakia

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ABSTRACT

A generalization of Picone's formula to the case of half-linear differential operators of the even order is given and comparison results concerning the associated differential equations are established with the help of this formula.

1. Introduction

The purpose of this work is to extend classical Sturm comparison theorems to half-linear differential equations of the form

$$\sum_{j=0}^{n}(-1)^j(p_j\varphi(u^{(j)}))^{(j)}=0, \tag{1}$$

and

$$\sum_{j=0}^{n}(-1)^j(P_j\varphi(v^{(j)}))^{(j)}=0 \tag{2}$$

where $n\geq 1$, p_j and P_j, $j=0,\ldots,n$, are continuous functions defined on $[a,b]$ and φ is a signed power function defined for given $\alpha>0$ by $\varphi(\xi):=|\xi|^\alpha \operatorname{sgn}\xi$, $\xi\in\mathbb{R}$. Our basic tool is a generalized Picone's identity established by the present author in [1].

Some weaker comparison results for (1) and (2) (in the sense that the Sturmian conclusion about zeros of a solution v (or its derivatives $v',\ldots,v^{(n-1)}$) of (2) applies to the closed interval $[a,b]$ rather than (a,b)) were obtained in [1]. In proving stronger results, we employ an adaptation of the method based on the introduction of an appropriate Sobolev space, used in the linear case $\alpha=1$ by Kusano and Yoshida in [2].

For related results concerning the special case $\alpha=1$ and general n see [3], and for the case $n=2$, also see [4–6]. Other comparison results for nonlinear equations can be found in [7,8].

2. The main results

Consider an even order nonlinear differential operator of the form

$$L_\alpha[y]\equiv\sum_{j=0}^{n}(-1)^j(P_j\varphi(y^{(j)}))^{(j)} \tag{3}$$

E-mail address: jaros@fmph.uniba.sk.

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where $n \geq 1$, $P_j, j = 0, \ldots, n$, are continuous functions defined on an interval I and $\varphi(\xi) := |\xi|^\alpha \operatorname{sgn} \xi, \alpha > 0, \xi \in \mathbb{R}$. Let $D_{L_\alpha}(I)$ denote the set of all continuous functions y defined on I such that y is n times continuously differentiable on I and $(P_j\varphi(y^{(j)}))^{(j)}, j = 1, \ldots, n$, exist and are continuous on I.

If we denote by Φ_α the form defined for $u, v \in \mathbb{R}$ and $\alpha > 0$ by

$$\Phi_\alpha(u, v) := u\varphi(u) + \alpha v\varphi(v) - (\alpha + 1)u\varphi(v), \tag{4}$$

then from the Young inequality it follows that $\Phi_\alpha(u, v) \geq 0$ for all $u, v \in \mathbb{R}$, and equality holds if and only if $u = v$.

The following lemma can be verified by direct computation.

Lemma 1 (*Weaker Form of Picone's Identity*). *If $x, y \in C^n(I)$, $P_j\varphi(y^{(j)}) \in C^j(I)$, $j = 0, \ldots, n$, and if none of $y, y', \ldots, y^{(n-1)}$ vanish in I, then*

$$\begin{aligned} &\frac{d}{dt}\left\{\sum_{k=0}^{n-1}(-1)^{k+1}\frac{|x^{(k)}|^{\alpha+1}}{\varphi(y^{(k)})}\sum_{j=k+1}^{n}(-1)^j(P_j\varphi(y^{(j)}))^{(j-k-1)}\right\} \\ &\quad= \frac{|x|^{\alpha+1}}{\varphi(y)}L_\alpha[y] + \sum_{j=0}^{n}P_j|x^{(j)}|^{\alpha+1} - P_n\Phi_\alpha\left(x^{(n)}, \frac{x^{(n-1)}y^{(n)}}{y^{(n-1)}}\right) \\ &\qquad - \sum_{k=1}^{n-1}\Phi_\alpha\left(\frac{x^{(k)}}{y^{(k)}}, \frac{x^{(k-1)}}{y^{(k-1)}}\right)(-1)^k y^{(k)}\sum_{j=k}^{n}(-1)^j(P_j\varphi(y^{(j)}))^{(j-k)}. \end{aligned} \tag{5}$$

As an application of the identity (5) we obtain the following variational result.

Theorem 1. *Suppose that $P_n(t) \geq 0$ in $[a, b]$ and there exists a nontrivial function $u \in C^n([a, b])$ such that*

$$u(a) = u'(a) = \cdots = u^{(n-1)}(a) = u(b) = \cdots = u^{(n-1)}(b) = 0 \tag{6}$$

and

$$M[u] \equiv \int_a^b \sum_{j=0}^{n} P_j(t)|u^{(j)}|^{\alpha+1}dt \leq 0. \tag{7}$$

If $v \in D_{L\alpha}([a, b])$ satisfies

$$vL_\alpha[v] \geq 0 \quad \text{in } (a, b), \tag{8}$$

$$v^{(k)}\sum_{j=k}^{n}(-1)^{j-k}(P_j(t)\varphi(v^{(j)}))^{(j-k)} \geq 0 \quad \text{in } (a, b),\ k = 1, \ldots, n-1, \tag{9}$$

and

$$\sum_{j=\nu}^{n}(-1)^j(P_j(t)\varphi(v^{(j)}))^{(j-\nu)} \neq 0 \quad \text{in } (a, b) \text{ for some } \nu \in \{1, \ldots, n-1\}, \tag{10}$$

then at least one of $v, v', \ldots, v^{(n-1)}$ has a zero in (a, b) unless u and v are linearly dependent in the case $n = 1$.

Proof. Suppose, to the contrary, that there exists a function $v \in D_{L_\alpha}([a, b])$ satisfying (8)–(10) in (a, b) for which none of $v, v', \ldots, v^{(n-1)}$ vanish in (a, b). We claim that in this case at least one of $v, v', \ldots, v^{(n-1)}$ must vanish at either $t = a$ or $t = b$. If not, then from the identity (5) with $x = u$ and $y = v$ integrated over $[a, b]$ we would have

$$\begin{aligned} 0 &\leq M[u] - \int_a^b\left\{\sum_{k=1}^{n-1}\Phi_\alpha\left(\frac{u^{(k)}}{v^{(k)}}, \frac{u^{(k-1)}}{v^{(k-1)}}\right)v^{(k)}\sum_{j=k}^{n}(-1)^{j-k}(P_j(t)\varphi(v^{(j)}))^{(j-k)}\right\}dt \\ &\leq -\int_a^b \Phi_\alpha\left(\frac{u^{(\nu)}}{v^{(\nu)}}, \frac{u^{(\nu-1)}}{v^{(\nu-1)}}\right)v^{(\nu)}\sum_{j=\nu}^{n}(-1)^{j-\nu}(P_j(t)\varphi_\alpha(v^{(j)}))^{(j-\nu)}dt \leq 0. \end{aligned}$$

Thus, we obtain

$$Q[u, v] \equiv \int_a^b \Phi_\alpha\left(\frac{u^{(\nu)}}{v^{(\nu)}}, \frac{u^{(\nu-1)}}{v^{(\nu-1)}}\right)v^{(\nu)}\sum_{j=\nu}^{n}(-1)^{j-\nu}(P_j(t)\varphi(v^{(j)}))^{(j-\nu)}dt = 0.$$

Condition (10) now implies that $u^{(\nu-1)} = cv^{(\nu-1)}$ on $[a, b]$ for some nonzero constant c which is a contradiction because u satisfies the boundary conditions (6) and $v^{(\nu-1)} \neq 0$ on $[a, b]$.

Further, the conditions imposed on u imply that u is a member of the Sobolev space $H_0^{n,\alpha+1}((a, b))$ which is the closure of $C_0^{\infty}((a, b))$ in the norm

$$\|w\| \equiv \left(\int_a^b \sum_{j=0}^{n} |w^{(j)}|^{\alpha+1} \right)^{1/(\alpha+1)}. \tag{11}$$

Let $\{u_m\}$ be a sequence of C^{∞} functions having compact support in (a, b) and converging to u as $m \to \infty$ in the norm (11). Then the identity (5) holds for $x = u_m$ and $y = v$, and like in the first part of the proof, we get $Q[u_m, v] = 0$.

Now, let J be a closed subinterval of (a, b) and define

$$Q_J[u_m, v] \equiv \int_J \Phi_\alpha\left(\frac{u_m^{(\nu)}}{v^{(\nu)}}, \frac{u_m^{(\nu-1)}}{v^{(\nu-1)}} \right) v^{(\nu)} \sum_{j=\nu}^{n} (-1)^{j-\nu} (P_j(t)\varphi(v^{(j)}))^{(j-\nu)} dt.$$

Using the Hölder inequality and the boundedness of $P_j, j = \nu, \ldots, n$, on J, we can verify that $Q_J[u_m, v] \to Q_J[u, v]$ as $m \to \infty$. Thus, we get $Q_J[u, v] = 0$, and since J is arbitrary, it follows that $u^{(\nu-1)}$ is a constant multiple of $v^{(\nu-1)}$. But this is a contradiction in the case $n \geq 2$, since $u^{(\nu-1)}(a) = 0 = u^{(\nu-1)}(b)$, while $v^{(\nu-1)}$ is strictly monotone on $[a, b]$. This completes the proof. □

By a reinterpretation of Theorem 1 we obtain another typical result that follows from the identity (5).

Corollary 1 (*Wirtinger-Type Inequality*). *If $P_n(t) \geq 0$ in $[a, b]$ and there exists $v \in D_{L_\alpha}([a, b])$ such that $vL_\alpha[v] \geq 0$, (9) and (10) are satisfied and none of $v, v', \ldots, v^{(n-1)}$ vanish in (a, b), then the inequality*

$$M[u] \equiv \int_a^b \sum_{j=0}^{n} P_j(t)|u^{(j)}|^{\alpha+1} dt \geq 0 \tag{12}$$

holds for any nontrivial function $u \in C^n([a, b])$ satisfying (6).

In addition to (3) consider now another half-linear differential operator

$$l_\alpha[x] \equiv \sum_{j=0}^{n} (-1)^j (p_j \varphi(x^{(j)}))^{(j)} \tag{13}$$

where $p_j, j = 0, \ldots, n$, are continuous functions on $[a, b]$, with the domain D_{l_α} defined analogously to D_{L_α}.

It is easy to verify that

$$\frac{d}{dt}\left\{ \sum_{k=0}^{n-1} (-1)^k x^{(k)} \sum_{j=k+1}^{n} (-1)^j (p_j \varphi(x^{(j)}))^{(j-k-1)} \right\} = x l_\alpha[x] - \sum_{j=0}^{n} p_j |x^{(j)}|^{\alpha+1}. \tag{14}$$

Combining (5) with (14) we get the following generalization of the classical Picone's formula.

Lemma 2 (*Stronger Form of Picone's Identity*). *If $x \in D_l(I)$, $y \in D_L(I)$, and if none of $y, y', \ldots, y^{(n-1)}$ vanish in I, then*

$$\begin{aligned} &\frac{d}{dt}\left\{ \sum_{k=0}^{n-1} \frac{(-1)^k x^{(k)}}{\varphi(y^{(k)})} \left[\varphi(x^{(k)}) \sum_{j=k+1}^{n} (-1)^j (P_j \varphi(y^{(j)}))^{(j-k-1)} - \varphi(y^{(k)}) \sum_{j=k+1}^{n} (-1)^j (p_j \varphi(x^{(j)}))^{(j-k-1)} \right] \right\} \\ &= \frac{x}{\varphi(y)} \{ \varphi(x) L_\alpha[y] - \varphi(y) l_\alpha[x] \} + \sum_{j=0}^{n} (p_j - P_j)|x^{(j)}|^{\alpha+1} + P_n \Phi_\alpha\left(x^{(n)}, \frac{x^{(n-1)} y^{(n)}}{y^{(n-1)}} \right) \\ &\quad + \sum_{k=1}^{n-1} \Phi_\alpha\left(\frac{x^{(k)}}{y^{(k)}}, \frac{x^{(k-1)}}{y^{(k-1)}} \right) (-1)^k y^{(k)} \sum_{j=k}^{n} (-1)^j (P_j \varphi(y^{(j)}))^{(j-k)}. \end{aligned} \tag{15}$$

On the basis of the identity (15) it is easy to prove the following comparison theorem.

Theorem 2. *Suppose that $P_n(t) \geq 0$ in $[a, b]$ and there exists a nontrivial function u in the domain $D_{l_\alpha}([a, b])$ of the operator l_α which satisfies*

$$\int_a^b u l_\alpha[u] dt \leq 0, \tag{16}$$

$$u(a) = u'(a) = \cdots = u^{(n-1)}(a) = u(b) = \cdots = u^{(n-1)}(b) = 0, \tag{17}$$

and

$$V[u] \equiv \int_a^b \sum_{j=0}^{n} [p_j(t) - P_j(t)]|u^{(j)}|^{\alpha+1} dt \geq 0. \tag{18}$$

Then for any $v \in D_{L_\alpha}([a, b])$ satisfying $vL_\alpha[v] \geq 0$ and (9)–(10) in (a, b) at least one of $v, v', \ldots, v^{(n-1)}$ must vanish in (a, b) unless u and v are linearly dependent in the case $n = 1$.

Proof. If we associate with l_α the $(\alpha + 1)$-degree functional m defined by

$$m[u] \equiv \int_a^b \sum_{j=0}^{n} p_j(t)|u^{(j)}|^{\alpha+1} dt, \tag{19}$$

then from the integrated form of (15) and the conditions imposed on u we obtain

$$V[u] = m[u] - M[u] \leq -M[u]$$

and the conclusion follows from Theorem 1.

Applying Theorem 2 to the particular case of two $2n$th-order equations of the form

$$(-1)^n (p_n(t)\varphi(u^{(n)}))^{(n)} + p_0(t)\varphi(u) = 0 \tag{20}$$

and

$$(-1)^n (P_n(t)\varphi(v^{(n)}))^{(n)} + P_0(t)\varphi(v) = 0, \tag{21}$$

where $n \geq 2$, $p_n(t) > 0$ and $P_n(t) > 0$ on $[a, b]$, we obtain the following comparison result. □

Corollary 2. *Let there exist a nontrivial solution u of Eq. (20) which satisfies (17) and*

$$\int_a^b \left\{[p_0(t) - P_0(t)]|u|^{\alpha+1} + [p_n(t) - P_n(t)]|u^{(n)}|^{\alpha+1}\right\} dt \geq 0. \tag{22}$$

Then for any solution v of (21) satisfying in (a, b)

$$v^{(n-1)}(P_n(t)\varphi(v^{(n)}))' \leq 0, \tag{23}$$

and

$$(P_n(t)\varphi(v^{(n)}))' \neq 0, \tag{24}$$

at least one of $v, v', \ldots, v^{(n-1)}$ must vanish in (a, b).

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